Sensitivity of a Class of Distributed Parameter Control Systems

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A sensitivity matrix is defined as a measure of trajectory deviations to small parameter variations of both open and closed loop controlled nonlinear parabolic and first-order hyperbolic systems. In general the parameters may enter through the system equations or the boundary conditions and may be time or spatially dependent. The introduction of a positive measure of the sensitivity, the norm of the sensitivity matrix, into the performance index is shown to be effective in limiting the trajectory deviations due to the parameter variations. The open and closed loop control of a double pipe heat exchanger is analyzed with the open loop problem solved by an approximate procedure. The sensitivity reformulation is successful in reducing trajectory sensitivity, however at the cost of decreased overall performance.

A basic premise of optimal control theory is the complete knowledge of the state equations governing the process. In most situations of practical interest, however, the mathematical model of the process is an idealization of the physical phenomena involved, and often the deviation of the actual performance from the performance predicted by the model is significant. An optimal or suboptimal control designed on the basis of a specific mathematical model may then yield unsatisfactory results in actual plant operation. It is thus of prime importance to be able to predict the effect of model inaccuracies on the control objective.

In a large number of cases the basic differential equations of the process have been established. Often, however, parameters entering into these equations are not exactly known, for example, transport coefficients in constitutive equations, catalyst activity, etc. The control scheme is normally designed on the basis of certain assumed values (nominal values) of these parameters, representing, perhaps, expected average values.

The study of the effect of parameter variations on the performace of lumped parameter optimal control systems was introduced by Dorato (*5*). Pagurek (*13*) and Witsenhausen (*18*) extended Dorato's analysis to linear systems with quadratic performance criteria and nonlinear systems, respectively. A significant result of Pagurek's analysis is that in many cases the open and closed loop sensitivities are identical for optimal systems. The linear-quadratic problem was treated further by Barnett (*1*). The concept of incorporating a sensitivity measure in the original performance index to form an augmented index on which to base control specifications was introduced by Cadzow (*2*) for linear-quadratic systems and subsequently extended to nonlinear systems by Dougherty, Lee, and De Russo (*6*).

Only recently has the optimal control of distributed parameter systems been placed theoretically and computationally in the wellknown lumped parameter format, that is, the maximum principle and dynamic programming (*7, 10, 17*). As in lumped parameter systems the ability to compensate for unknown variations in a distributed control system is essential for a thorough analysis of a practical situation. We will concentrate on parabolic and first-order hyperbolic systems, the classes of which include a large number of common processes of practical interest. The extension of the analysis to second-order hyperbolic and elliptic systems is straight forward.

We assume that nominal values of certain parameters in the system model are known, but the exact value is unknown. We assume thus that the parameter values are deterministic, but undergo variations and at any time have unknown exact values. In the main development we will consider constant parameters, indicating briefly the analysis for time-varying or spatially-varying (or both) parameters.

We treat first the case of sensitivity of optimal open loop controls, which we represent most generally by considering the deviation in the state trajectory, since the change in all other quantities of interest, that is, performance index, terminal and trajectory constraints, etc., can ultimately be derived from the trajectory deviation. Secondly we consider the class of distributed systems for which an optimal closed loop law is realizable. We show that such optimal laws cannot in general be obtained. Then we consider the class of suboptimal closed loop controls where the functional form of the feedback law has been specified a priori and an optimal gain matrix must be determined.

In each case we adopt the concept of incorporating a sensitivity measure in the performance index to limit the trajectory dispersion from the nominal parameter values. The relative importance of optimality in the sense of the performance index and reduced sensitivity is adjusted by selection of an appropriate weighting matrix in the sensitivity norm. Sensitivity will be defined in the classic sense as the normalized change in the desired quantity divided by the normalized change of the variable parameter. We might have employed one of the alternative definitions of sensitivity that have been proposed (*15*), however, once the basic sensitivity information is known, its interpretation depends on the particular situation and purpose.

SENSITIVITY OF THE OPEN LOOP CONTROL

Consider a distributed parameter system defined on a fixed spatial domain Ω, an open, simply connected, bounded subset of the N dimensional Euclidean space E_N. Let the boundary of Ω be denoted by $\partial\Omega$ and the range of the independent variable $t, (0, t_f)$, be denoted by Γ. The systems of interest are assumed to be described by the parabolic or first-order hyperbolic partial differential equation and boundary conditions,

$$\mathbf{u}_t(t, \mathbf{x}) = \mathbf{f}[t, \mathbf{x}, \mathbf{u}(t, \mathbf{x}), \mathbf{u}_\mathbf{x}(t, \mathbf{x}), \mathbf{u}_{\mathbf{xx}}(t, \mathbf{x}),$$

$$\mathbf{v}(t,\mathbf{x}),\mathbf{a}(t,\mathbf{x})] \text{ in } \Omega \times \Gamma \quad (1)$$

$$\boldsymbol{\zeta}_o[\mathbf{u}(o,\mathbf{x})] = \mathbf{0}$$

$$\boldsymbol{\zeta}[t,\mathbf{x},\mathbf{u}(t,\mathbf{x}),\mathbf{u}_x(t,\mathbf{x}),\mathbf{w}(t),\mathbf{b}(t)] = \mathbf{0} \quad \mathbf{x}\epsilon\partial\Omega$$

where $\mathbf{u}(t,\mathbf{x}) = [u_1(t,\mathbf{x}), u_2(t,\mathbf{x}), \ldots, u_n(t,\mathbf{x})]$ is the state of the system and $\mathbf{v}(t,\mathbf{x}) = [v_1(t,\mathbf{x}), v_2(t,\mathbf{x}), \ldots, v_m(t,\mathbf{x})]$ and $\mathbf{w}(t) = [w_1(t), w_2(t), \ldots, w_q(t)]$ are distributed and boundary controls, allowed to assume values in $L_2(\Omega \times \Gamma)$ and $L_2(\partial\Omega \times \Gamma)$, respectively, where L_2 is the space of Lebesque square integrable functions. Specifically, $\mathbf{v}(t,\mathbf{x})$ and $\mathbf{w}(t)$ are piece-by-piece continuous functions of their arguments and take values from bounded convex regions V and W. $\mathbf{a}(t,\mathbf{x}) = [a_1(t,\mathbf{x}), a_2(t,\mathbf{x}), \ldots, a_r(t,\mathbf{x})]$ and $\mathbf{b}(t) = [b_1(t), b_2(t), \ldots, b_s(t)]$ are parameters in the system equations and boundary conditions, respectively. $\mathbf{f}$ is assumed to have continuous first-order derivatives with respect to $\mathbf{x}$ and t and is twice continuously differentiable with respect to the remaining arguments.

The optimal open loop control problem consists of determining $\mathbf{v}(t,\mathbf{x})$ and $\mathbf{w}(t)$, $\mathbf{v}(t,\mathbf{x}) \epsilon V$, $\mathbf{w}(t) \epsilon W$, to minimize a scalar functional $P[\mathbf{u}(t,\mathbf{x}),\mathbf{v}(t,\mathbf{x}),\mathbf{w}(t)]$ dependent in general on the system and control trajectories.

The optimal control problem may be solved assuming $\mathbf{a}(t,\mathbf{x})$ and $\mathbf{b}(t)$ are at some nominal values $\mathbf{a}_o(t,\mathbf{x})$ and $\mathbf{b}_o(t)$ to yield the open loop control laws,

$$\begin{aligned} \mathbf{v}(t,\mathbf{x}) &= \boldsymbol{\eta}(t,\mathbf{x}) \\ \mathbf{w}(t) &= \boldsymbol{\theta}(t) \end{aligned} \quad (2)$$

to which corresponds the state trajectory,

$$\mathbf{u}(t,\mathbf{x}) = \boldsymbol{\psi}[t,\mathbf{x},\boldsymbol{\eta}(t,\mathbf{x}),\boldsymbol{\theta}(t),\mathbf{a}_o(t,\mathbf{x}),\mathbf{b}_o(t)] \quad (3)$$

In general, the actual parameter values are $\mathbf{a}(t,\mathbf{x})$ and $\mathbf{b}(t)$, different from the nominal values $\mathbf{a}_o(t,\mathbf{x})$ and $\mathbf{b}_o(t)$. The control laws, Equation (2), then become nonoptimal for the system at the new parameter values. We desire first to predict the alteration of the system trajectory, Equation (3), when $\mathbf{a}(t,\mathbf{x})$ and $\mathbf{b}(t)$ assume values other than the nominal $\mathbf{a}_o(t,\mathbf{x})$ and $\mathbf{b}_o(t)$. For simplicity in the following we will assume a single spatial variable x and distributed control $\boldsymbol{\eta}(t,x)$ only. The extension to higher spatial dimensions and to simultaneous boundary and distributed control is straightforward.

The nominal state trajectory is thus expressed by

$$\mathbf{u}(t,x) = \boldsymbol{\psi}[t,x,\boldsymbol{\eta}(t,x),\mathbf{a}_o(t,x),\mathbf{b}_o(t)] \quad (4)$$

and the actual trajectory given by

$$\mathbf{u}(t,x) = \boldsymbol{\psi}[t,x,\boldsymbol{\eta}(t,x),\mathbf{a}(t,x),\mathbf{b}(t)] \quad (5)$$

The deviation due to parameter variation is then

$$\Delta\mathbf{u}(t,x) = \boldsymbol{\psi}[t,x,\boldsymbol{\eta}(t,x),\mathbf{a}(t,x),\mathbf{b}(t)] - \boldsymbol{\psi}[t,x,\boldsymbol{\eta}(t,x)\,\mathbf{a}_o(t,x),\mathbf{b}_o(t) \quad (6)$$

Consider first the case where $\mathbf{a}(t,x)$ and $\mathbf{b}(t)$ represent vectors of constant parameters, $\mathbf{a} = (a_1, a_2, \ldots, a_r)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_s)$. For sufficiently small variations in $\mathbf{a}$ and $\mathbf{b}$ we may rewrite Equation (6) as

$$\Delta\mathbf{u}(t,x) = \boldsymbol{\psi}_a\,(\mathbf{a}-\mathbf{a}_o) + \boldsymbol{\psi}_b\,(\mathbf{b}-\mathbf{b}_o) + \ldots \quad (7)$$

where $\boldsymbol{\psi}_a = [\partial\boldsymbol{\psi}/\partial\mathbf{a}]_{a_o,b_o}$ and $\boldsymbol{\psi}_b = [\partial\boldsymbol{\psi}/\partial\mathbf{b}]_{a_o,b_o}$. We may define the $n \times r$ and $n \times s$ open loop sensitivity matrices $\boldsymbol{\Lambda}^a$ and $\boldsymbol{\Lambda}^b$ by $\boldsymbol{\Lambda}^a = \boldsymbol{\psi}_a$ and $\boldsymbol{\Lambda}^b = \boldsymbol{\psi}_b$. $\boldsymbol{\Lambda}^a$ and $\boldsymbol{\Lambda}^b$ are a direct measure of the absolute sensitivity of the system trajectory to small perturbations in the parameters $\mathbf{a}$ and $\mathbf{b}$. Since $\boldsymbol{\Lambda}^a$ and $\boldsymbol{\Lambda}^b$ are defined for the entire trajectory, deviations of the performance functional $P[\mathbf{u}(t,x), \mathbf{v}(t,x)]$ or of terminal state values $\mathbf{u}(t_f,x)$ may be derived readily from $\boldsymbol{\Lambda}^a$ and $\boldsymbol{\Lambda}^b$.

From the definition of $\boldsymbol{\psi}[t,x,\boldsymbol{\eta}(t,x),\mathbf{a},\mathbf{b}]$ we note that

$$\boldsymbol{\psi}_t[t,x,\boldsymbol{\eta}(t,x),\mathbf{a},\mathbf{b}] = \mathbf{f}[t,x,\mathbf{u}(t,x),\mathbf{u}_x(t,x),\mathbf{u}_{xx}(t,x),\boldsymbol{\eta}(t,x),\mathbf{a},\mathbf{b}] \quad (8)$$

Provided we can reverse the order of differentiation in $\partial/\partial t\ \partial\boldsymbol{\psi}/\partial\mathbf{a}$ and $\partial/\partial\mathbf{a}\ \partial\boldsymbol{\psi}/\partial t$ we obtain the evolution equations for $\boldsymbol{\Lambda}^a$ and $\boldsymbol{\Lambda}^b$,

$$\boldsymbol{\Lambda}^a{}_t = \mathbf{f}_u\,\boldsymbol{\Lambda}^a + \mathbf{f}_{u_x}\,\boldsymbol{\Lambda}^a{}_x + \mathbf{f}_{u_{xx}}\,\boldsymbol{\Lambda}^a{}_{xx} + \mathbf{f}_a \text{ in } \Omega \times \Gamma \quad (9)$$

and

$$\boldsymbol{\Lambda}^b{}_t = \mathbf{f}_u\,\boldsymbol{\Lambda}^b + \mathbf{f}_{u_x}\,\boldsymbol{\Lambda}^b{}_x + \mathbf{f}_{u_{xx}}\,\boldsymbol{\Lambda}^b{}_{xx} \text{ in } \Omega \times \Gamma \quad (10)$$

subject to the boundary conditions

$$\boldsymbol{\zeta}_u\,\boldsymbol{\Lambda}^a + \boldsymbol{\zeta}_{u_x}\,\boldsymbol{\Lambda}^a{}_x = \mathbf{N} \quad x \epsilon \partial\Omega \quad (11)$$

$$\boldsymbol{\zeta}_u\,\boldsymbol{\Lambda}^b + \boldsymbol{\zeta}_{u_x}\,\boldsymbol{\Lambda}^b{}_x + \boldsymbol{\zeta}_b = \mathbf{N} \quad x \epsilon \partial\Omega \quad (12)$$

where $\mathbf{N}$ is the null matrix. The initial conditions for Equations (8) and (9) are

$$\boldsymbol{\Lambda}^a = \mathbf{N}, \quad t = 0 \quad (13)$$

and

$$\boldsymbol{\Lambda}^b = \mathbf{N}, \quad t = 0$$

since the initial state $\boldsymbol{\zeta}_0[\mathbf{u}(o,x)]$ was assumed independent of $\mathbf{a}$ and $\mathbf{b}$.

For the case when $\mathbf{a} = \mathbf{a}(t,x)$ and $\mathbf{b} = \mathbf{b}(t)$, Equation (7) becomes

$$\Delta\mathbf{u}(t,x) = \left[\frac{\partial\boldsymbol{\psi}}{\partial\mathbf{a}}\right]_{a_o(t,x),b_o(t)} [\mathbf{a}(t,x) - \mathbf{a}_o(t,x)] + \left[\frac{\partial\boldsymbol{\psi}}{\partial\mathbf{b}}\right]_{a_o(t,x),b_o(t)} [\mathbf{b}(t) - \mathbf{b}_o(t)] + \ldots \quad (14)$$

One may define the *sensitivity matrices* $\boldsymbol{\Lambda}^a$ and $\boldsymbol{\Lambda}^b$ in terms of the above partial derivatives and proceed as before. A somewhat easier approach has been suggested by Dougherty, et. al. (*6*). Consider the variational equations of (1), namely

$$\delta\mathbf{u}_t(t,x) = \mathbf{f}_u\,\delta\mathbf{u}(t,x) + \mathbf{f}_{u_x}\,\delta\mathbf{u}_x(t,x) + \mathbf{f}_{u_{xx}}\,\delta\mathbf{u}_{xx}(t,x) + \mathbf{f}_a\delta_a(t,x) \text{ in } \Omega \times \Gamma$$

$$\delta\mathbf{u}(o,x) = \mathbf{0}$$

$$\boldsymbol{\zeta}_u\,\delta\mathbf{u}(t,x) + \boldsymbol{\zeta}_{u_x}\,\delta\mathbf{u}_x(t,x) + \boldsymbol{\zeta}_b\,\delta\mathbf{b}(t) = \mathbf{0} \quad x \epsilon \partial\Omega \quad (15)$$

If it is possible to ascribe a maximum bound to $|\delta\mathbf{a}(t,x)|$ and $|\delta\mathbf{b}(t,x)|$ we may solve Equation (15) to determine the corresponding trajectory deviation $\delta\mathbf{u}(t,x)$.

If it is desired to limit the trajectory deviation as a result of parameter variations, a new optimal control problem can be formulated in which an augmented performance functional J,

$$J = P[\mathbf{u}(t,x),\mathbf{v}(t,x)] + P'\,[||\boldsymbol{\Lambda}^a(t,x)||, ||\boldsymbol{\Lambda}^b(t,x)||] \quad (16)$$

is formed which includes a term depending on the norms of $\boldsymbol{\Lambda}^a$ and $\boldsymbol{\Lambda}^b$ in addition to the original index P. By selection of arbitrary weighting matrices in $||\boldsymbol{\Lambda}^a||$ and $||\boldsymbol{\Lambda}^b||$ we are able to affect a trade-off between the original control criterion P and the desired reduction in trajectory sensitivity. The sensitivity reduction problem is absorbed into the optimal control problem in much the same way as state and control constraints are often formulated in terms of a penalty function.

EXAMPLE

As an example we consider a concentric double pipe heat exchanger with an incompressible fluid on the tube side and a condensing vapor on the shell side. In particular, we assume:

1. Constant specific heat and density of the incompressible fluid.
2. No variation of vapor temperature with length in the exchanger.
3. Negligible axial heat flow in the fluid and walls.
4. Negligible outer pipe dynamics.

The fluid temperature $T_1(t', x')$ and the wall temperature $T_2(t', x')$ are then governed by the two first order hyperbolic equations

$$\rho_1 c_{p1}\ \pi R_1^2[T_{1t'}(t', x') + \nu_1 T_{1x'}(t', x')] = h_{21} 2\pi R_1[T_2(t', x') - T_1(t', x')]$$

$$\rho_2 c_{p2}\ \pi(R_2^2 - R_1^2)\ T_{2t'}(t', x') = h_{32}\ 2\pi R_2[y(t') - T_2(t', x')] - h_{21}\ 2\pi R_1[T_2(t', x') - T_1(t', x')] \quad (17)$$

with initial conditions,

$$T_1(0, x') = T_{1ss}(x'), \quad T_2(0, x') = T_{2ss}(x'), \quad y(0) = y_o \quad (18)$$

and boundary condition,

$$T_1(t', o) = T_{1_o}(t') \quad (19)$$

where $y(t')$ is the temperature of the condensing vapor. Let us define the dimensionless variables,

$$t = \frac{\nu_1 t'}{L'}$$

$$x = \frac{x'}{L'}$$

$$c_1 = \frac{2h_{21}L'}{R_1\ \nu_1\rho_1 c_{p1}}$$

$$c_2 = \frac{2h_{32}R_2L'}{(R_2^2 - R_1^2)\ \nu_1\rho_2 c_{p2}}$$

$$c_3 = \frac{2h_{21}R_1L'}{(R_2^2 - R_1^2)\ \nu_1\rho_2 c_{p2}} \quad (20)$$

We pose the problem of choosing $y(t)$ subject to the physical constraints, $y^* \geqq y(t) \geqq y_*$, to drive the temperature of the outlet fluid, $T_1(t, 1)$, from its initial value, $T_{1ss}(1)$, as close as possible to a new desired value, $T_1^d(1)$, corresponding to the new steady state, $T_1^d(x)$, $T_2^d(x)$, and y_d in a fixed time t_f. We thus wish to minimize

$$P = \int_o^{t_f} [T_1(t, 1) - T_1^d(1)]^2\ dt \quad (21)$$

If we introduce the normalized variables,

$$u_1(t, x) = 2[T_1(t, x) - T_1^d(x)]/(y^* - y_*)$$
$$u_2(t, x) = 2[T_2(t, x) - T_2^d(x)]/(y^* - y_*)$$
$$v(t) = 2[y(t) - y_d]/(y^* - y_*) \quad (22)$$

the desired values of $u_1(t, x)$, $u_2(t, x)$, and $v(t)$ are zero and the system equations and boundary conditions are

$$u_{1t}(t, x) + u_{1x}(t, x) = c_1[u_2(t, x) - u_1(t, x)]$$
$$u_{2t}(t, x) = c_2[v(t) - u_2(t, x)] - c_3[u_2(t, x) - u_1(t, x)]$$
$$u_1(0, x) = u_{1ss}(x),\ u_2(0, x) = u_{2ss}(x),\ v(0) = v_o$$
$$u_1(t, 0) = 0 \quad (23)$$

and the performance index we desire to minimize by choice of $v(t)$ is

$$P = \int_o^{t_f} u_1(t, 1)^2\ dt \quad (24)$$

The bounds on the transformed control, $v^* \geqq v(t) \geqq v_*$ are determined from

$$v^* = 2(y_d - y_*)/(y^* - y_*)$$
$$v_* = 2[1 - (y_d - y_*)]/(y^* - y_*) \quad (25)$$

The parameters c_1, c_2, and c_3 depend on the heat transfer coefficients h_{21} and h_{32}, as well as the fluid velocity ν_1, density ρ_1, and specific heat c_{p1}. In solving the optimal control problem we have posed, we must assume definite values for c_1, c_2, and c_3. However, in an actual plant situation unknown fluctuations upstream may cause any of these quantities to vary, causing the state equations to deviate from those used as a basis for selecting the vapor temperature policy. In particular, let us consider the sensitivity of the system to variations in the shell side heat transfer coefficient h_{32}, which are manifested by variations in c_2.

Since only a scalar parameter is involved we define a sensitivity vector $\boldsymbol{\lambda}$ with components $\lambda_1(t, x)$ and $\lambda_2(t, x)$ defined by

$$\lambda_1(t, x) = \left[\frac{\partial\psi_1[t, x, v(t),\ c_1, c_2, c_3]}{\partial c_2}\right]_{c_{2_o}}$$

$$\lambda_2(t, x) = \left[\frac{\partial\psi_2[t, x, v(t),\ c_1, c_2, c_3]}{\partial c_2}\right]_{c_{2_o}} \quad (26)$$

where $u_1(t, x) = \psi_1[t, x, v(t), c_1, c_2, c_3]$ and $u_2(t, x) = \psi[t, x, v(t), c_1, c_2, c_3]$ and c_{2_o} represents the nominal value of c_2.

It is easily shown that for this example, Equation (9) reduces to the two first-order hyperbolic equations,

$$\lambda_{1t}(t, x) + \lambda_{1x}(t, x) = c_1[\lambda_2(t, x) - \lambda_1(t, x)]$$
$$\lambda_{2t}(t, x) = c_3\lambda_1(t, x) - (c_{2_o} + c_3)\lambda_2(t, x) + v(t) - u_2(t, x) \quad (27)$$

with boundary conditions,

$$\lambda_1(o, x) = 0, \quad \lambda_2(o, x) = 0$$
$$\lambda_1(t, o) = 0 \quad (28)$$

We now pose a new optimal control problem by forming the augmented performance index, $J = P + P'$, where

$$P = \int_o^{t_f} u_1(t, 1)^2\ dt$$

$$P' = \int_o^{t_f} \boldsymbol{\lambda}^T(t, 1)\mathbf{Q}\ \boldsymbol{\lambda}(t, 1)\ dt$$

We desire to determine $v(t)$, $v^* \geqq v(t) \geqq v_*$, to minimize J subject to Equations (23), (27), and (28). $\mathbf{Q}$ is a positive semidefinite symmetric matrix, the elements of which govern the degree of weighting in the new overall performance index, J, between P and P'.

Nominal parameter values were taken from the work of Cohen (*3*) and Lesser (*12*):

$$c_1 = 1.29 \qquad v^* = 2/3$$
$$c_{2_o} = 5.50 \qquad v_* = -4/3$$
$$c_3 = 2.52 \qquad v_o = -2/3$$

We will assume a diagonal form for $\mathbf{Q}$ so that P' becomes

$$P' = \int_o^{t_f} \{q_1\ \lambda_1(t, 1)^2 + q_2\ \lambda_2^2(t, 1)\}\ dt \quad (29)$$

The necessary conditions for optimality in the augmented control problem can be obtained from the maximum principle for distributed parameter systems, the elements of which are given in detail elsewhere (*16*). We will assume a familiarity with the concepts and present briefly the necessary conditions for our example.

If we define an additional state variable $u_3(t)$ by

$$u_{3_t}(t) = -[u_1^2(t,1) + q_1 \lambda_1^2(t,1) + q_2 \lambda_2^2(t,1)]$$
$$u_3(0) = 0 \tag{30}$$

the performance objective becomes $J = -u_3(t_f)$.

We define a Hamiltonian by

$$\begin{aligned} H = {} & p_1[-u_{1_x} + c_1(u_2 - u_1)] \\ & + p_2[c_2(v - u_2) - c_3(u_2 - u_1)] \\ & + p_3[-(u_1^2 + q_1 \lambda_1^2 + q_2 \lambda_2^2)] \\ & + p_4[-\lambda_{1_x} + c_1(\lambda_2 - \lambda_1)] \\ & + p_5[c_3 \lambda_1 - (c_2 + c_3)\lambda_2 + v - u_2] \end{aligned} \tag{31}$$

where the adjoint vector $\mathbf{p}(t,x) = [p_1(t,x), p_2(t,x), \ldots, p_5(t,x)]$ is governed by

$$\begin{aligned} & p_{1t} + p_{1x} = c_1 p_1 - c_3 p_2 + 2u_1 \, p_3 \\ & p_{2t} = -c_1 p_1 + (c_2 + c_3) p_2 + p_5 \\ & p_{3t} = 0 \\ & p_{4t} + p_{4x} = c_1 p_4 - c_3 p_5 + 2q_1 \lambda_1 \, p_3 \\ & p_{5t} = -c_1 p_4 + (c_2 + c_3) p_5 + 2q_2 \lambda_2 \, p_3 \\ & p_1(t_f, x) = 0 \qquad p_1(t, 1) = 0 \\ & p_2(t_f, x) = 0 \\ & p_3(t_f, x) = \delta(x - 1) \\ & p_4(t_f, x) = 0 \qquad p_4(t, 1) = 0 \\ & p_5(t_f, x) = 0 \end{aligned} \tag{32}$$

The necessary condition for optimality is that $\int_0^1 H\,dx$ be a maximum with respect to $v(t)$. The optimal control is clearly given by

$$v(t) = \begin{cases} v^* & \int_0^1 \{c_2 p_2(t,x) + p_5(t,x)\}\,dx > 0 \\ v_* & \int_0^1 \{c_2 p_2(t,x) + p_5(t,x)\}\,dx < 0 \end{cases} \tag{33}$$

The system defined by Equations (23), (27), (28), (30), (32), and (33) represents a two point boundary-value problem in functional space of fairly significant dimensionality (ten variables). Two basic computational methods have recently been developed for solution of problems of this type: a method of steepest ascent in functional space due to Denn, et al. (*4*), Jackson (*9*), and Seinfeld (*16*) and a direct search on the performance index due to Seinfeld (*16*). We have chosen to use the latter method for this example. Specific comparisons of advantages, computation requirements, etc., of the two methods appear elsewhere (*16*).

The elements of the direct search method are as follows. We assume that the interval $(0, t_f)$ is divided into L segments and $(0, 1)$ divided into R segments. We select M discrete admissible values of $v(t)$. The solution of Equation (23) can then be represented by $(L + 1) \times (R + 1)$ mesh points. We develop the mesh values for known $v(i)$, $i = 1, 2, \ldots, L - 1$, by an appropriate numerical method for solving partial differential equations. The performance index J is evaluated on the $i = L$ row of the mesh. The direct search algorithm can be outlined as follows,

1. Guess $v(i)$, $i = 1, 2, \ldots, L - 1$. Integrate the system (23), (27), and (28) from $i = 0$ to $i = L$ and evaluate the performance index, denoted by $J^{(o)}$.
2. For each of the M admissible values of $v(Q)$ integrate the system from $i = 0$ to $i = L$ and evaluate J. Retain that value of $v(o)$ that yields the smallest value of J, which we denote as $J^{(1)}$. [Note that the worst we can do is retain the original $v(o)$].
3. Repeat the procedure for $v(i)$, $i = 1, 2, \ldots, L - 1$, until new values are developed for all $v(i)$. In each iteration $J^{(k+1)} \geqq J^{(k)}$.
4. Return to $i = 0$ and repeat the overall procedure until two successive overall iterations do not improve the value of J.

A particular advantage of this method is the elimination of the formalism of the maximum principle, although the result is only locally optimal for the specific L, R, and M chosen.

The response of the outlet fluid $u_1(t, 1)$ can be obtained analytically, in particular (*3*),

$$\begin{aligned} u_1(t,1) = {} & u_{1_{ss}}(1) + \left(1 + \frac{r_2}{r_1 - r_2} e^{r_1 t} + \frac{r_1}{r_2 - r_1} e^{r_2 t}\right) \\ & - \hat{h}(t-1) r_1 r_2 \int_0^{t-1} \frac{e^{r_1(t-\tau)} - e^{r_2(t-\tau)}}{(r_1 - r_2)} \\ & \cdot \left(1 + \int_0^{\tau} e^{-(c_2+c_3)\mu} \frac{c_1 c_2^{1/2}}{\mu} I_1[2(c_1 c_2 \mu)]^{1/2}\,d\mu\right) d\tau \end{aligned} \tag{34}$$

where $\hat{h}$ is the Heaviside unit step function and r_1 and r_2 are the roots of

$$\frac{s^2}{c_1 c_2} + \left(\frac{1}{c_1} + \frac{1}{c_2} + \frac{c_3}{c_1 c_2}\right) s + 1 = 0 \tag{35}$$

From Equation (34) we see that for $t \leqq 1$ the dynamics of the outlet fluid are governed by the second order ordinary differential equation,

$$\frac{d^2}{dt^2} u_1(t,1) - (r_1 + r_2)\frac{du_1(t,1)}{dt} + r_1 r_2 u_1(t,1) = r_1 r_2[v(t) - \kappa] \tag{36}$$

where $\kappa = v_o - u_{1_{ss}}(1)$. Only after $t = 1$ does the distributed nature of the heat exchange become apparent. We may expect for the original optimization problem one switch from v^* to v_* followed by a singular interval of $v(t) = -0.276$ maintaining $u_1(t, 1)$ at zero, provided, of course, that the origin can be reached by $t \leqq 1$.

We will use the more general approach of the direct search with $t_f = 1$ and 2 since in the case of variable parameters, a time-optimal $t_f \leqq 1$ *cannot be established a priori* and solutions for $t_f > 1$ are of interest. In each case studied an optimization problem was solved with and without the additional sensitivity penalty P' in the total performance index J. In seven equally spaced control values, $-4/3$, -1, $-2/3$, $\ldots$, $2/3$ were used with $L = 21$ and $R = 21$ for the computation. Different values of the weighting factors q_1 and q_2 were used to examine the effect of an increased sensitivity penalty on the control policy. A variety of initial control policy guesses were tried with each specific case and no essential differences resulted in the final control policies. About 17 sec. per overall iteration were required for $M = 7$, $L = 21$, and $R = 21$ on an IBM 7094 computer.

The results in Table 1 were obtained based on $c_{2_0} = 5.50$. It is obvious that as the sensitivity weighting factors

q_1 and q_2 are increased the original performance index P increases, since the resulting control policy is chosen to accommodate the augmented index.

TABLE 1.

Case	t_f	q_1	q_2	overall iterations to convergence	P	P'
1	1	0	0	3	0.0357	0
2	1	1.0	1.0	5	0.0359	0.0021
3	1	5.0	5.0	5	0.0371	0.0087
4	1	25.0	25.0	4	0.0514	0.0186
5	2	0	0	7	0.0451	0
6	2	25.0	25.0	5	0.0663	0.0381

TABLE 2.

control policy from case	P	ΔP	$\Delta P/P\%$
1	0.0360	0.0003	0.84
4	0.0517	0.0003	0.58
5	0.0454	0.0003	0.66
6	0.0663	<0.0001	0

Next c_2 was set equal to 6.0, as opposed to the nominal value of 5.50. For this new condition the control policies from cases 1 and 4 ($t_f = 1$) and 5 and 6 ($t_f = 2$) were compared to evaluate the effectiveness of the control policies determined with a nonzero P' (that is, cases 4 and 6) in reducing the sensitivity of P to the variation in c_2. The results for $c_2 = 6.0$ are shown in Table 2.

The first significant observation is that for the specific parameter values employed, the outlet fluid response is fairly insensitive to variations in c_2. The variation of c_2 from 5.50 to 6.0 produced only 0.84% increase in the performance index under the control from case 1, in Figure 1, determined without regard to the system sensitivity. When the control policy from case 4 also shown in Figure 1 was used, the absolute sensitivity was reduced to 0.58%, with a proportionate rise in P from 0.0360 to 0.0517. For the case $t_f = 2$, the control from case 5 produced an absolute sensitivity in P of 0.66%, whereas the policy from case 6 reduced the sensitivity essentially to zero, however with a sacrifice in the original objective of 0.0454 to 0.0663.

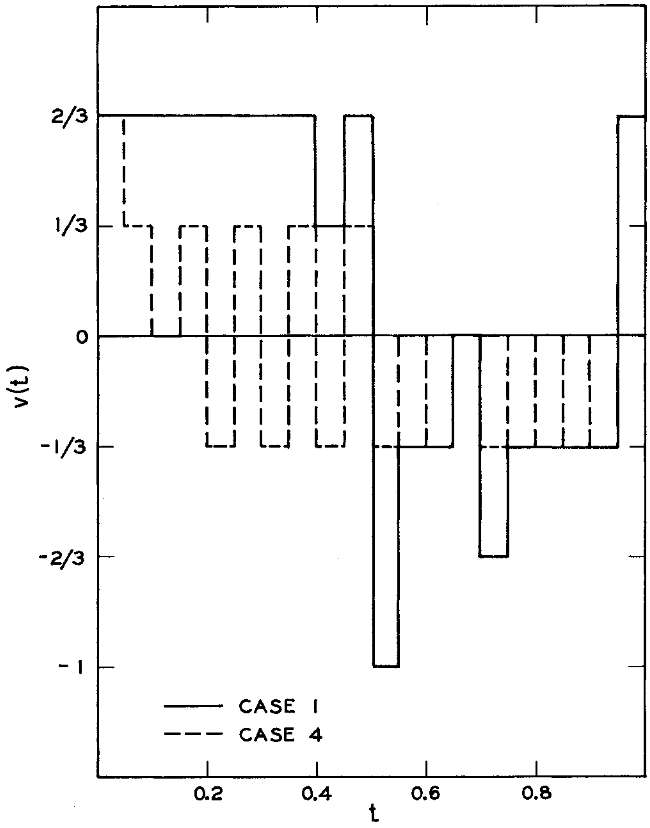

Fig. 1. Comparison of control policies for heat exchanger.

Thus, if the original performance index, Equation (24) is deemed by far the more important criterion, it is advisable for this system to disregard sensitivity considerations. However, if the speed of the outlet fluid response is to be sacrificed to reduce the sensitivity of the outlet fluid to parameter fluctuations, a policy from cases 2, 3, 4, or 6 should be selected. Cases were also run with $c_2 = 5.0$ and essentially the same results were obtained. However, in this instance, P was decreased below its optimal value for $c_2 = 5.50$, which, of course, is always liable to occur in the presence of unknown variations.

To test the sensitivity of the system over a range of parameter values, c_1 and c_3 were increased by a factor of ten to 12.9 and 25.2, respectively, while retaining c_2 at its nominal value of 5.5. Such an increase in c_1 and c_3 might be experienced if the heat transfer coefficient h_{21} were increased by a factor of ten in some manner. The results of the direct search solution to the control problem are shown in Table 3. Table 4 shows the results obtained when the control policies from cases 7 and 8 were used with $c_2 = 6.0$.

It is again obvious that the tenfold reduction in sensitivity obtained in case 8 is gained only at the expense of an increase in P. In general, the performance index (24) is relatively insensitive to control variations, indicating a flat performance surface P. Other distributed systems could undoubtedly exhibit more significant sensitivity effects.

TABLE 3.

Case	t_f	q_1	q_2	overall iterations to convergence	P	P'
7	1	0	0	5	0.0570	0
8	1	25	25	6	0.0757	0.0523

TABLE 4.

control policy from case	P	ΔP	$\Delta P/P\%$
7	0.0578	0.0008	1.4
8	0.0758	0.0001	0.13

SENSITIVITY OF THE CLOSING LOOP CONTROL

The control law obtained from the solution of the preceding optimization problem is necessarily in an open loop form, for example, Equation (33). Only in quite special cases can an optimal closed loop law be constructed. The most wellknown example is the linear lumped parameter system with a quadratic performance index, in which the canonical equations may be uncoupled by the Ricatti transformation to yield an optimal feedback control law. Let us briefly consider the question of constructing an optimal feedback control law for distributed systems. Wang and Tung (*17*) derived an optimal feedback law for distributed control of a slab with a temperature distribution governed by the inhomogeneous heat equation and a quadratic performance index. Their method was extended by Erzberger and Kim (*8*) to include the boundary control of systems described by the homogeneous heat equation. Both of these classes can be combined to examine the feasibility of deriving optimal closed loop control laws in a distributed system.

Assume that the solution of Equation (1) can be represented by the set of integral equations,

$$\mathbf{u}(t, x) = \int_\Gamma \int_\Omega \mathbf{K}[t, \tau, x, \mu, \ \mathbf{u}(\tau,\mu)\ \mathbf{v}(\tau,\mu), \mathbf{w}(\tau), \mathbf{a}(\tau, \mu), \mathbf{b}(\tau)]\, d\mu d\tau \quad (37)$$

where $\mathbf{K}(\ldots)$ is the customary Hilbert-Schmidt kernel. We express the general performance index as

$$P = \int_0^{t_f} \gamma_1(t) dt + \gamma_2(t_f) \quad (38)$$

Let the minimum of P with respect to $\mathbf{v}(t, x)$ and $\mathbf{w}(t)$ be given by

$$\Phi[\mathbf{u}(t, x), t_f - t] = \min_{v,w} \left\{ \int_t^{t_f} \gamma_1(t) dt + \gamma_2(t_f) \right\} \quad (39)$$

subject to the final condition $\Phi[\mathbf{u}(t_f, x), o] = \gamma_2(t_f)$.

Applying the principle of optimality,

$$\Phi[\mathbf{u}(t, x), t_f - t] = \min_{v,w} \left\{ \Phi[\mathbf{u}(t + \Delta t, x), t_f - t - \Delta t] + \int_t^{t+\Delta t} \gamma_1(t) dt \right\} \quad (40)$$

If we assume that the solution of the system equations with initial state $\mathbf{u}(t, x)$ and sufficiently small Δt can be approximated by

$$\mathbf{u}(t + \Delta t, x) = \mathbf{u}(t, x) + \Delta t \frac{\partial}{\partial t} \int_o^t \int_\Omega \mathbf{K}(\ldots)\, d\mu\, d\tau + \ldots \quad (41)$$

we can expand the functional $\Phi[\mathbf{u}(t + \Delta t, x), t_f - t - \Delta t]$ about $\mathbf{u}(t, x)$ and $t_f - t$ to obtain

$$\Phi[\mathbf{u}(t + \Delta t, x), t_f - t - \Delta t] = \Phi[\mathbf{u}(t, + \Delta t, x), t_f - t] + \Delta t \frac{\partial \Phi}{\partial (t_f - t)} + \frac{\delta \Phi}{\delta \mathbf{u}} \frac{\partial}{\partial t} \int_o^t \int_\Omega \mathbf{K}(\ldots) d\mu d\tau \Big] dx \quad (42)$$

where $\delta\Phi/\delta\mathbf{u}$ is a functional partial derivative, defined as the variation of the functional $\Phi[\mathbf{u}(t, x), t_f - t]$ due to a variation in the value of the function $\mathbf{u}(t, x)$ at a specific point $x \in \Omega$ (*17*). As $\Delta t \to 0$, Equation (40) becomes

$$\frac{\partial \Phi}{\partial (t_f - t)} = \min_{v,w} \int_\Omega \left[\frac{\delta \Phi}{\delta \mathbf{u}} \frac{\partial}{\partial t} \int_o^t \int_\Omega \mathbf{K}(\ldots)\, d\mu d\tau \right] dx + \gamma_1(t) \quad (43)$$

a functional form of the Hamilton-Jacobi partial differential equation. When the performance index P is quadratic in $\mathbf{u}(t,x)$, $\mathbf{v}(t, x)$, and $\mathbf{w}(t)$, we may assume a solution of Equation (8) in the form (15)

$$\Phi[\mathbf{u}(t, x), t_f - t] = \frac{1}{2} \int_\Omega \int_\Omega \mathbf{u}(t, \mu_1) \, \boldsymbol{\Theta}\, (\mu_1, \mu_2, t_f - t)\, \mathbf{u}(t, \mu_2) d\mu_1, d\mu_2 \quad (44)$$

yielding a nonlinear partial differential integral Ricatti equation for $\boldsymbol{\Theta}(\mu_1, \mu_2, t_f - t)$. Solutions to such equations can only be obtained in the simplest of cases, for example, the one dimensional heat Equation (15). It is thus not possible to generate optimal feedback controls for any of but a few idealized distributed systems. If a closed loop scheme is required it will then in almost all cases be suboptimal with regard to the open loop optimal control formulation.

The analysis of the sensitivity of a closed loop control law will be similar to the open loop case. It is important to add that the design of closed loop controls for a nonlinear distributed system is by no means a trivial question. In fact, to the author's knowledge, there exist few published results on this important aspect of automatic control theory (*14*). For the purposes of this study we will assume that a closed loop law has been determined by some appropriate approximation.

Given the form of a closed loop control law $\mathbf{v}(t, x) = \boldsymbol{\beta}[t, x, \mathbf{u}(t, x), \mathbf{k}]$ containing an adjustable gain vector $\mathbf{k}$ we pose the same problem as above, except that the only choice variable is $\mathbf{k}$. The problem reduces to one of multiparameter optimization on the k_i to minimize P. The evolution Equations (9) and (10) now become

$$\boldsymbol{\Lambda}^a{}_t = \mathbf{f_u}\, \boldsymbol{\Lambda}^a + \mathbf{f}_{\mathbf{u}_x} \boldsymbol{\Lambda}^a{}_x + \mathbf{f}_{\mathbf{u}_{xx}} \boldsymbol{\Lambda}^a{}_{xx} + \mathbf{f_a} + \mathbf{f}_\beta\, \boldsymbol{\beta}_\mathbf{u}\, \boldsymbol{\Lambda}^a \quad (45)$$

and

$$\boldsymbol{\Lambda}^b{}_t = \mathbf{f_u}\, \boldsymbol{\Lambda}^b + \mathbf{f}_{\mathbf{u}_x} \boldsymbol{\Lambda}^b{}_x + \mathbf{f}_{\mathbf{u}_{xx}} \boldsymbol{\Lambda}^b{}_{xx} + \mathbf{f}_\beta\, \boldsymbol{\beta}_\mathbf{u}\, \boldsymbol{\Lambda}^b \quad (46)$$

subject to Equations (11), (12), and (13).

The parameter optimization problem may be formulated as before with an augmented performance index.

As a particular example, consider the heat exchanger. Rather than attempt to solve the optimal control problem we specify *a priori* a control law of the form $v(t) = -k\, u_1(t, x)\, \delta(x - 1)$. The state Equations (23) become

$$u_{1t}(t, x) + u_{1x}(t, x) = c_1[u_2(t, x) - u_1(t, x)]$$

$$u_{2t}(t, x) = (c_3 - c_2) u_2(t, x) + u_1(t, x)[c_3 - kc_2\ \delta(x - 1)] \quad (47)$$

and the sensitivity vector $\lambda(t, x) = [\lambda_1(t, x), \lambda_2(t, x)]$ is

$$\lambda_{1t}(t, x) + \lambda_{1x}(t, x) = c_1[\lambda_2(t, x) - \lambda_1(t, x)]$$

$$\lambda_{2t}(t, x) = [c_3 - k\, c_2 \delta(x - 1)]\, \lambda_1(t, x) - (c_2 + c_3)\lambda_2(t, x) - u_x(t, x) \quad (48)$$

The problem is simply to select k to minimize J, which can be solved as a one parameter optimization.

The preceding analysis of closed loop sensitivity raises a question of more general interest, namely, other than in the special linear-quadratic case, to what extent can optimization theory be used to design closed loop controls. In the absence of variations the open and closed loop performances are identical. When uncertainty exists, the performance index value depends not only on the values of the variable parameters but also on the functional form of the control law. It is necessary to devise a way of comparing controllers in the presence of uncertainty.

The feedback control problem exists only if the environment is subject to uncertainty and the open loop optimization problem exists in principle only if the environment is certain. One approach to reconciling this inherent difference is through formulation of the so-called "inverse" problem, whereby a control law is specified *a priori* and the class of systems for which the given law is optimal are determined.

It is important to note that the above analysis has been based on small deviations of the parameters about nominal values. In a practical situation, parameter variations may not be small when compared to the nominal values. In addition, the nominal values themselves may be unknown, only the limits of variation of the parameters being available. One approach would be to assign a probability distribution function to $\mathbf{a}$ and $\mathbf{b}$ and formulate a stochastic control problem. A more practical way of treating the closed loop problem is through a min-max formulation (11). Given $\mathbf{v}(t, x) = \boldsymbol{\beta}[t, x, \mathbf{u}(t, x), \mathbf{k}]$, we desire to choose $\mathbf{k}$ so that regardless of what value the parameters, say, $\mathbf{a}$, assume, the performance of the system approximates optimal open loop behavior. Specifically, we want to compute

$$\underset{\mathbf{k}}{\text{Min}} \quad \underset{\mathbf{a}}{\text{Max}} \left\{ \frac{P(\mathbf{k}, \mathbf{a}) - P_{\text{opt}}(\mathbf{a})}{P_{\text{opt}}(\mathbf{a})} \right\}$$

The obvious drawback of this formulation for distributed parameter systems is the computation required to implement the algorithm.

CONCLUSIONS

The sensitivity of both open and closed loop distributed control systems has been investigated and a technique for reducing the effects of parameter variations by augmenting the performance index demonstrated. In the open loop case one can solve a new optimization problem with the augmented performance index and additional sensitivity Equations (9) and (10).

The formulation is successful in decreasing the sensitivity of the state trajectories in a double-pipe heat exchanger, however at the expense of a proportionate increase in the original performance index. The problem thus presents a trade-off to the designer between the original optimality criterion and reduced trajectory dispersion due to parameter variations.

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NOTATION

$\mathbf{a}(t, x)$ = parameter vector in system model
$\mathbf{b}(t)$ = parameter vector in boundary conditions
c_1, c_2, c_3 = parameters in double-pipe heat exchanger, defined in Equation (20)
c_{p1}, c_{p2} = specific heats of fluid and wall respectively
h_{21}, h_{32} = heat transfer coefficients
H = Hamiltonian function
J = performance functional
$\mathbf{k}$ = gain vector
$\mathbf{K}$ = Hilbert-Schmidt kernel in Equation (37)
L = mesh points in t direction
L' = length of heat exchanger
M = discrete control choices
m = dimension of distributed control vector $\mathbf{v}(t, x)$
$\mathbf{N}$ = null matrix
n = dimension of state vector $\mathbf{u}(t, x)$
$\mathbf{O}$ = null vector
$p_1, \ldots, p_5$ = adjoint variables
P = performance functional
q_1, q_2 = elements of $\mathbf{Q}$
q = dimension of boundary control vector $\mathbf{w}(t)$
$\mathbf{Q}$ = weighting matrix
r = dimension of $\mathbf{a}(t, x)$
r_1, r_2 = roots of Equation (35)
R = mesh points in x direction
R_1 = inner radius of inner pipe
R_2 = outer radius of inner pipe
s = dimension of $\mathbf{b}(t)$, Laplace transform variable
t, t' = time variables
T_1, T_2 = temperatures
$\mathbf{u}(t, x)$ = state vector
$\mathbf{v}(t, x)$ = distributed control vector
$\mathbf{w}(t)$ = boundary control vector
x, x' = spatial variables
y = steam temperature

Greek Letters

$\boldsymbol{\beta}$ = closed loop distributed control vector
γ_1, γ_2 = performance criteria in Equation (38)
Γ = range of independent variable t
δ = small increment
Δ = difference operator
$\boldsymbol{\zeta}$ = boundary condition vector
$\boldsymbol{\eta}$ = optimal open loop distributed control vector
$\boldsymbol{\theta}$ = optimal open loop boundary control vector
κ = constant in Equation (36)
$\boldsymbol{\Lambda}^a$, $\boldsymbol{\Lambda}^b$ = sensitivity matrices, $n \times r$ and $n \times s$, respectively
$\boldsymbol{\lambda}$ = sensitivity vector
μ_1, μ_2 = spatial variables
ν_1 = fluid velocity
ρ_1, ρ_2 = fluid and wall densities
τ = integration variable
Φ = minimum value of the performance functional P
$\boldsymbol{\Psi}$ = state trajectory
Ω = spatial domain of distributed system

Superscripts

d = desired
$*$ = upper bound

Subscripts

o = nominal or initial
ss = steady state
d = desired
$*$ = lower bound

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